

Analogues of the van der Waerden and Tverberg conjectures for hafnians

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Abstract

We discuss here analogues of van der Waerden and Tverberg permanent conjectures for hafnians on the convex set of matrices whose extreme points are symmetric permutation matrices with zero diagonal.

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1 Introduction

Let $G = (V, E)$ be an undirected graph with no loops but possibly with multi-edges. Denote by $\mathcal{M}_k(G)$ the subset of all k -matchings. It is well known that for a given graph G there exist polynomial time algorithms to find if $\mathcal{M}_k(G)$ is empty or not [4]. However, even for bipartite graph the problem of finding $\#\mathcal{M}_k(G)$ is #P-complete problem, see [25] for perfect matching and [18] for k -matchings. There are fully randomized polynomial approximation schemes to estimate $\#\mathcal{M}_k(G)$ in bipartite graphs [21, 18].

The aim of this paper is to discuss lower bounds for $\#\mathcal{M}_k(G)$ for regular graphs. A lot is known for regular bipartite graphs. This follows from the validity of the van der Waerden and Tverberg conjectures for permanents of doubly stochastic matrices [27, 24, 11, 6, 9, 12]. Improved lower bounds were obtained for regular bipartite graphs [26, 23, 20, 15]. One of the main features of regular bipartite graphs that they can be expressed as a k edge-disjoint union of perfect matches. More precisely the Birkhoff theorem yields that any doubly stochastic matrix is a convex combination of permutation matrices.

For regular nonbipartite graph the situation is more complex. There exist simple cubic graphs which do not have a perfect matching. See for example the Sylvester graph [22, Figure P.3, p'xii]. The celebrated Petersen's theorem claims that a cubic graph with two bridges at most has a perfect matching. It was only shown very recently that a simple cubic graph without a bridge have exponential number of perfect matchings in the number of vertices [8]. Let $G = (V, E)$ be an undirected

graph with no loops and denote by $A(G)$ the adjacency matrix of G . Then $A(G)$ is a symmetric matrix with zero diagonal and nonnegative integer off-diagonal entries. For a set $T \subset \mathbb{R}$ denote by $S_0(n, T)$ the set of symmetric matrices with zero diagonal whose off-diagonal entries are in T . Thus, $B = [b_{ij}] \in S_0(n, \mathbb{R}_+)$ can be viewed as viewed as a weighted complete graph K_n , where $0 \leq b_{ij}$ is the weight of the edge (i, j) . Assume that $n = 2m$ is even. Denote by $\text{haf}(B)$, the *haffnian* of B , the sum of the weighted perfected matches of K_n , given by B . For any n and positive integer k , such that $2k \leq n$, and $B \in S_0(n, \mathbb{R}_+)$, denote by $\text{haf}_k(B)$ the sum of all $2k$ haffnians of principle submatrices of order $2k$ of B . Equivalently, $\text{haf}_k(B)$ is the sum of all weighted k -matches in K_n given by B . Thus for a given graph $G = (V, E)$, $\text{haf}_k(A(G))$ is the number of k -matches in G .

Suppose that $G = (V, E)$ is bipartite, where $V = V_1 \cup V_2, E \subset V_1 \times V_2$. Then $\hat{A}(G) \in \mathbb{Z}_+^{|V_1| \times |V_2|}$ is the bipartite adjacency matrix of G , i.e. $A(G) = \begin{bmatrix} 0 & \hat{A}(G) \\ (\hat{A}(G))^\top & 0 \end{bmatrix}$. More generally, given positive integers l, m and an $l \times m$ nonnegative matrix $C = [c_{ij}] \in \mathbb{R}_+^{l \times m}$, then C can be viewed as a weighted bipartite graph $K_{l,m}$. Thus $B = \begin{bmatrix} 0 & C \\ C^\top & 0 \end{bmatrix}$ be the adjacency matrix of the weighted graph $K_{l,m}$. Then $\text{haf}_k(B) = \text{perm}_k(C)$, where $\text{perm}_k(C)$ is the sum of permanents of all $k \times k$ submatrices of C . Denote by $\Omega_n \subset \mathbb{R}_+^{n \times n}$ the set of doubly stochastic matrices. The proved Tverberg conjecture states [12].

$$\min_{C \in \Omega_n} \text{perm}_k(C) = \text{perm}_k\left(\frac{1}{n} \hat{A}(K_{n,n})\right) \text{ for } k = 2, \dots, n. \quad (1.1)$$

Equality holds if and only if $C = \frac{1}{n} \hat{A}(K_{n,n})$. The case $k = n$ is the van der Waerden conjecture. (1.1) for $k = n$ implies immediately that any r -regular bipartite graph on $2n$ vertices has at least $(\frac{r}{e})^n$ perfect matchings. For better bounds see [26, 23, 20].

The main success in proving the Tverberg conjecture and its sharper analogs for r -regular bipartite graphs can be attributed to the notion of *hyperbolicity*. Suffices to say that a product of linear factors is hyperbolic. Hence for any nonnegative matrix $C \in \mathbb{R}_+^{n \times n}$ the polynomial $f(\mathbf{x}) := \prod_{i=1}^n (C\mathbf{x})_i$ is positive hyperbolic. ($f(\mathbf{x})$ is a sum of monomial with nonnegative coefficients.) Furthermore $\text{perm}(C) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} f$, is the mixed derivative of f .

The aim of this paper is to introduce analogous problems to the van der Waerden and Tverberg conjectures. First, one needs to introduce an analog notion to the notion of doubly stochastic matrices $\Psi_{2n} \subset S(2n, \mathbb{R}_+) \cap \Omega_{2n}$. Namely, this is the convex set of symmetric doubly stochastic matrices with zero diagonal, whose extreme points are symmetric permutation matrices with zero diagonal. Ψ_{2n} was characterized by Edmonds [5], see [2, Theorem 6.3, 2nd Proof, page 209] for a simple proof. Namely, it is the set of all $(2n) \times (2n)$ stochastic matrices $B = [b_{ij}] \in \Omega_{2n}$, which are symmetric and have zero diagonal, that satisfy the condition

$$\sum_{i,j \in S} b_{ij} \leq |S| - 1, \text{ for each } S \subset 1, \dots, 2n, |S| \text{ odd and } 3 \leq |S| \leq 2n - 3. \quad (1.2)$$

Our problem is to find or give a good lower bound for

$$\min_{B \in \Psi_{2n}} \text{haf}_k(B) = \mu_{k,n} \text{ for } k = 2, \dots, n. \quad (1.3)$$

It is tempting to state, as in the case of the van der Waerden and Tverberg's conjectures that

$$\mu_{k,n} = \text{haf}_k\left(\frac{1}{2n-1}A(K_{2n})\right) \text{ for } k = 2, \dots, n. \quad (1.4)$$

Equality holds if and only if $B = \frac{1}{2n-1}A(K_{2n})$. See [13]. (According to a recent e-mail from Leonid Gurvits, he stated this conjecture for $k = n$ in correspondence with E. Lieb on September 21, 2005.) As we show in §3, this conjecture is true for $k = 2$. However for $k = n$ and n big enough (1.4) is wrong as explained below.

Note that if $G = (V, E)$ is a r -regular graph without loops on an even number of vertices, then $\frac{1}{r}A(G)$ is in $\Psi_{|V|}$ if and only if any vertex cut $S \subset V$ with an odd number of vertices, $3 \leq |S| \leq |V| - 3$, has at least r edges. Hence if Conjecture (1.4) holds, then such a regular graph has at least $\left(\frac{r}{e}\right)^{\frac{|V|}{2}}$ perfect matchings, see (2.4). In [3] the authors construct an infinite family of 3-edge connected graph $G = (V, E)$, i.e. an edge disjoint union of 3-perfect matchings, for which the number of matchings is less than $c_F|V|\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{|V|}{12}}$. (Here $|V| = 12k + 4$ and $k = 1, 2, \dots$) As $\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{12}} < 1.017 < \sqrt{\frac{3}{e}} \approx 1.05$ we must have that $\mu_{n,n} < \text{haf}\left(\frac{1}{2n-1}A(K_{2n})\right)$ for $n \gg 1$. (I would like to thank S. Norin for pointing out to me this fact.)

Since $\mu_{n,n}$ is the minimum of the haffnian function, it follows that $\mu_{n+m,n+m} \leq \mu_{n,n}\mu_{m,m}$. Hence the sequence $\log \mu_{n,n}$ is subadditive. In particular the following limit exists

$$\mu := \lim_{n \rightarrow \infty} \frac{\log \mu_{n,n}}{n}. \quad (1.5)$$

A weak analog of the van der Waerden conjecture is the claim that $\mu > -\infty$. Note that the above example in [3] implies that $\mu \leq \frac{\log \frac{1+\sqrt{5}}{2}}{6} - \log 3$. Other generalizations of the van der Waerden conjectures for perfect matchings in hypergraphs are considered in [1].

In §2 we estimate evaluate $\text{haf}_k\left(\frac{1}{2n-1}A(K_{2n})\right)$ and estimate the value of $\text{haf}_n\left(\frac{1}{2n-1}A(K_{2n})\right)$ for large n . Using the notion of hyperbolicity we give a good lower bound of $\text{haf}_k(B)$ for $B \in \Psi_{2n}$ with one positive eigenvalue. In §3 we show that $\frac{1}{2n-1}A(K_{2n})$ is a strict local minimum of $\text{haf}_k(\cdot)$ on Ψ_{2n} for $k = 2, \dots, n$.

2 Some equalities and lower estimates

Lemma 2.1 *For positive integers $2 \leq k \leq n$ we have*

$$\text{haf}_k\left(\frac{1}{2n-1}A(K_{2n})\right) = \frac{1}{(2n-1)^k} \binom{2n}{2k} \frac{1}{k!} \prod_{j=0}^{k-1} \binom{2k-2j}{2}, \quad (2.1)$$

$$\text{haf}_k\left(\frac{1}{n}A(K_{n,n})\right) = \frac{1}{n^k} \binom{n}{k}^2 k!. \quad (2.2)$$

In particular

$$e^{-n}\sqrt{2} < \text{haf}_n\left(\frac{1}{2n-1}A(K_{2n})\right) = \frac{(2n)!}{(2n-1)^n 2^n n!} < \text{haf}_n\left(\frac{1}{n}A(K_{n,n})\right) = \frac{n!}{n^n} \quad (2.3)$$

For $n \gg 1$ we have the following approximations

$$\text{haf}_n\left(\frac{1}{2n-1}A(K_{2n})\right) \approx e^{-n}\sqrt{2e}, \quad \text{haf}_n\left(\frac{1}{n}A(K_{n,n})\right) \approx e^{-n}\sqrt{2\pi n}. \quad (2.4)$$

Proof. We first compute the number of perfect matchings in K_{2n} . For $l = 1, \dots, n$ choose the l -th match between two distinct vertices of K_{2n} and remove these two vertices from the vertices of K_{2n} . Then number of choices for the l -th pair is $\binom{2n-2(l-1)}{2}$. Hence the total number of choices of n pairs, taking in account the order of choices is $\prod_{j=0}^{n-1} \binom{2n-2j}{2}$. Hence

$$\text{haf}_n(A(K_{2n})) = \frac{\prod_{j=0}^{n-1} \binom{2n-2j}{2}}{n!} = \frac{(2n)!}{2^n n!}.$$

Observe next that to find all k matchings in K_{2n} we first choose $2k$ vertices out of $2n$ vertices in K_{2n} , to obtain a subgraph K_{2k} of K_{2n} . Then we compute all perfect matchings in K_{2k} . So $\text{haf}_k(A(K_{2n})) = \binom{2n}{2k} \text{haf}_k(A(K_{2k}))$. This proves (2.1). (2.2) is well known [12], and proved similarly.

To obtain the first inequality in (2.3) we need the exact form of Stirling's formula [10, p.52].

$$m! = \sqrt{2\pi m} m^m e^{-m} e^{\frac{\theta_m}{12m}}, \quad \theta_m \in (0, 1), \quad m = 1, 2, \dots \quad (2.5)$$

Hence

$$\frac{(2n)!}{(2n-1)^n 2^n n!} \geq \frac{\sqrt{2\pi 2n} (2n)^{2n} e^n}{e^{2n} (2n-1)^n 2^n \sqrt{2\pi n} n^n e^{\frac{1}{12}}} = e^{-n} \sqrt{2} \left(\frac{2n}{2n-1}\right)^n e^{-\frac{1}{12}}.$$

Recall that the sequence $(\frac{m}{m-1})^{m-1}$ is an increasing sequence for $m = 2, \dots$. Hence $(\frac{2n}{2n-1})^{n-\frac{1}{2}}$ is an increasing sequence. As $n \geq 2$ we deduce that the left hand-side of the above inequality is greater than $e^{-n} \sqrt{2} (\frac{4}{3})^{\frac{3}{2}} e^{-\frac{1}{12}} > e^{-n} \sqrt{2}$. This establishes the left-hand side of (2.3). To establish the second inequality of (2.3) divide the middle expression of (2.3) by its right-hand side to obtain $(\frac{2n}{2n-1})^n 4^{-n} \binom{2n}{n}$. Recall that the sequence $(\frac{m}{m-1})^m$ is decreasing for $m = 2, \dots$. Hence the sequence $(\frac{2n}{2n-1})^n$ is a decreasing sequence. Hence to show the second inequality of (2.3) one needs to show that $(\frac{4}{3})^2 4^{-n} \binom{2n}{n} < 1$ for $n \geq 2$. This claim follows easily by induction. (2.4) follows straightforward from Stirling's formula. \square

Theorem 2.2 Let $B \in \Psi_{2n}$. Assume that $B \in \Psi_{2n}$ has exactly one positive eigenvalue. Then

$$\text{haf}_n(B) \geq \left(\frac{n-1}{n}\right)^{(n-1)n} \approx e^{-n} \sqrt{e}. \quad (2.6)$$

Moreover, for each $k = 2, \dots, n-1$

$$\text{haf}_k(B) \geq \frac{(2n)^{2n-2k} (2n-k)! (2n)^k}{(2n-2k)! (2n-k)^{2n-k} 2^k k!} \left(\frac{2n-k-1}{2n-k}\right)^{(2n-k-1)k}. \quad (2.7)$$

Proof. The above inequalities follows from the results of [14] as follows. It is well known that the quadratic polynomial $\mathbf{x}^\top B \mathbf{x}$ is hyperbolic, if and only if it has exactly one positive eigenvalue, e.g. [14, Lemma 6.1]. Observe next that

$$\text{Cap}((\mathbf{x}^\top B \mathbf{x})^k) := \inf_{\mathbf{x}=(x_1, \dots, x_{2n})^\top > \mathbf{0}} \frac{(\mathbf{x}^\top B \mathbf{x})^k}{\left(\prod_{i=1}^{2n} x_i\right)^{\frac{k}{n}}} = (\mathbf{1}^\top B \mathbf{1})^k = (2n)^k \quad (2.8)$$

for each $B \in \Psi_{2n}$. Here $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^{2n}$. In view of arithmetic-geometric inequality we deduce that above equality if B corresponds to a symmetric permutation matrix with zero diagonal. Hence the above equality holds each $B \in \Psi_{2n}$. Recall that

$$\text{haf}_k C = (2^k k!)^{-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq 2n} \frac{\partial^{2k}}{\partial x_{i_1} \dots \partial x_{i_{2k}}} (\mathbf{x}^\top C \mathbf{x})^k, \quad C \in S_m(0, \mathbb{R}_+), \quad (2.9)$$

where $2k \leq m$. See [14, §6]. Apply [14, Theorem 3.1] to the hyperbolic polynomial $\frac{(\mathbf{x}^\top B \mathbf{x})^n}{2^n n!}$ to estimate its mixed derivative with respect to all $2n$ variables to deduce the inequality in (2.6). (Replace n by $2n$, note that $r_1 = \dots = r_{2n} = n$, $k = n+1$ and use the equality (2.8).) Use the approximate expansion $\log(1-x) = -x - \frac{x^2}{2} + O(x^3)$ to deduce the approximation in (2.6) for $n \gg 1$. (2.7) is deduced similarly from [14, Theorem 3.1]. \square

3 Local conditions

Lemma 3.1 *Let $2 \leq k \leq n$ be integers. Then $C = \frac{1}{2n-1} A(K_{2n})$ is an interior point of the convex set Ψ_{2n} . C is a critical point of $\text{haf}_k(\cdot)$ on Ψ_{2n} . The hessian of $\text{haf}_k(\cdot)$ at C has positive eigenvalues. Hence C is a unique local minimum of $\text{haf}_k(\cdot)$. In particular, Conjecture 1.4 holds for $k = 2$.*

Proof. Let $S \subset \{1, \dots, 2n\}$. Then the sum of all the entries of C for $i, j \in S$ is $\frac{|S|(|S|-1)}{2n-1}$, which is strictly less than $|S| - 1$ if $|S| \in [2, 2n-2]$. Hence C is an interior point Ψ_{2n} . More precisely, each point $X \in \Psi_{2n}$ in the neighborhood of C if and only if it is of the form $C + Y$, where $Y = [y_{ij}]$ belongs to the subspace

$$\Phi_{2n} = \{Y \in S_0(2n, \mathbb{R}), Y\mathbf{1} = \mathbf{0}\}. \quad (3.1)$$

Observe next that for each integer $k \in [2, n]$ we have the equality

$$\text{haf}_k(F) = \frac{1}{k} \sum_{i < j} f_{ij} \text{haf}_{k-1}(F[[2n] \setminus \{i, j\}]), \quad F = [f_{ij}] \in S_0(2n, \mathbb{R}). \quad (3.2)$$

Here for any positive integer l we let $[l] = \{1, \dots, l\}$. Furthermore, for any $X = [x_{ij}] \in \mathbb{R}^{l \times l}$ and any $S \subseteq [l]$ we denote by $X[S]$ the principle submatrix $[x_{ij}]_{i,j \in S}$.

Hence

$$\begin{aligned} \text{haf}_k(C + Y) &= \text{haf}_k(C) + \frac{1}{2k} \sum_{i \neq j} y_{ij} \frac{1}{(2n-1)^{k-1}} \text{haf}_{k-1}(A(K_{2n-2})) + \\ &\frac{2}{k(k-1)} \sum_{1 \leq i < j < p < q \leq 2n, |S|=4} (y_{ij}y_{pq} + y_{ip}y_{jq} + y_{iq}y_{jp}) \frac{1}{(2n-1)^{k-2}} h_{k-2}(A(K_{2n-4})) + O(\|X\|^3). \end{aligned}$$

Here $\text{haf}_0(F) = 1$ for any $F \in S_0(2n, \mathbb{R})$. Since $Y\mathbf{1} = \mathbf{0}$ the linear term in the above expression is identically zero. Hence C is a critical point of $\text{haf}_k(\cdot)$ on Ψ_{2n} . Observe next that for $X = [x_{ij}] \in S_0(2n, \mathbb{R})$

$$2 \sum_{1 \leq i < j < p < q \leq 2n} (x_{ij}x_{pq} + x_{ip}x_{jq} + x_{iq}x_{jp}) = \left(\sum_{1 \leq i < j \leq 2n} x_{ij} \right)^2 - \sum_{i=1}^{2n} \left(\sum_{j=1}^{2n} x_{ij} \right)^2 + \sum_{1 \leq i < j \leq 2n} x_{ij}^2.$$

Hence for $Y \in \Phi_{2n}$ we have the equality

$$\sum_{1 \leq i < j < p < q \leq 2n} (y_{ij}y_{pq} + y_{ip}x_{jq} + y_{iq}y_{jp}) = \frac{1}{2} \sum_{1 \leq i < j \leq 2n} y_{ij}^2.$$

So the Hessian of $\text{haf}_k(\cdot)$ at C on Ψ_{2n} is strictly positive definite.

Let $k = 2$. Since $\text{haf}_2(C + Y)$ is a degree 2 polynomial the identity

$$\text{haf}_2(C + Y) = \text{haf}_2(C) + \frac{1}{2} \sum_{1 \leq i < j \leq 2n} y_{ij}^2 > \text{haf}_2(C)$$

if $Y \in \Phi_{2n} \setminus \{0\}$. As $C + \Phi_{2n} \supset \Psi_{2n}$ we deduce that Conjecture 1.4 holds for $k = 2$. \square

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